Lecture 21.

Product measures. We shall define the product measure of 2 measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$. An inductive procedure works for n spaces but we shall restrict to $n=2$.

Recall. On $X \times Y$, we have introduced the product σ-algebra $\mathcal{M} \otimes \mathcal{N}$. In the σ-finite case, it is generated by $A \times B$, $A \in \mathcal{M}$, $B \in \mathcal{N}$. We shall define $\mu \times \nu$ on $\mathcal{M} \otimes \mathcal{N}$ by first defining a premeasure π on an algebra $\mathcal{A}$ that generates $\mathcal{M} \otimes \mathcal{N}$.

(1) The collection of rectangles $A \times B$ as above is an elementary family,

Prop 1.7 $\Rightarrow$ $\mathcal{A} = \{$finite disjoint unions of such$\}$ is an algebra.

② Given disjoint union $\dot{\bigcup}_{k=1}^{n} A_k \times B_k$ in $\mathcal{A}$, let

$$\pi\left(\dot{\bigcup}_{k=1}^{n} A_k \times B_k\right) = \sum_{k=1}^{n} \mu(A_k)\nu(B_k).$$

Then, π is well-defined on $\mathcal{A}$ (check diff. representation gives same result; cf. before) and π is a premeasure: Note that if $\dot{\bigcup}_{k=1}^{\infty} A_k \times B_k$ a disjoint union in $\mathcal{A}$

then $\varphi_n(x,y) = \sum_{k=1}^{n} \chi_{A_k}(x)\chi_{B_k}(y)$ is

a simple function separately in x, y, in $L^+(X)$, $L^+(Y)$, and $\varphi_n \nearrow \varphi = \sum_{k=1}^{\infty} \chi_{A_k}\chi_{B_k}$.

Since $\dot{\bigcup}_{1}^{\infty} A_k \times B_k$ in $\mathcal{A}$ it is a <u>finite</u> disjoint union of rectangles. Assume, for simplicity that it is just $A \times B$ ($m=1$; the general case can be reduced to this).

Then, $\chi_{A\times B}(x,y) = \sum_{k=1}^{\infty} \chi_{A_k}(x)\chi_{B_k}(y)$

Also, $\chi_{A\times B}(x,y) = \chi_A(x)\,\chi_B(y)$.

Integrating in x, by MCT $\Rightarrow$

$$\mu(A)\chi_B(y) = \sum_{k=1}^{\infty} \mu(A_k)\,\chi_{B_k}(y)$$

Integrating in y $\Rightarrow$

$$\mu(A)\nu(B) = \sum_{k=1}^{\infty} \mu(A_k)\,\nu(B_k)$$

Thus,

$$\pi(A\times B) = \mu(A)\nu(B) = \sum_{k=1}^{\infty} \pi(A_k\times B_k).$$

$\Rightarrow$ π is premeasure on $\mathcal{A}$.

By "standard procedure", $\pi \to$ outer measure $\to$ measure on $\mathcal{M}\otimes\mathcal{N}$ which coincides w/ π on $\mathcal{A}$.

Def. This is the product measure $\mu\times\nu$ on $\mathcal{M}\otimes\mathcal{N}$.

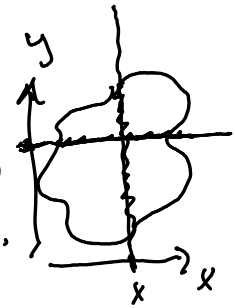
Rem. When μ, ν are σ-finite, so is $\mu \times \nu$. In this case, $\mu \times \nu$ is the unique measure on $\mathcal{M} \otimes \mathcal{N}$ that coincides w/ π on $\mathcal{A}$.

General sets. Let $E \in \mathcal{M} \otimes \mathcal{N}$.

Define x- and y-sections by

$$E_x = \{y \in Y : (x,y) \in E\}$$

$$E^y = \{x \in X : (x,y) \in E\}.$$



For $f : X \times Y \to \mathbb{C}$ (or $\mathbb{R}$ or ...)

let

$$f_x(y) = f(x,y) = f^y(x).$$

Prop 1. $E \in \mathcal{M} \otimes \mathcal{N} \Rightarrow E_x \in \mathcal{N}, E^y \in \mathcal{M}$ for all x, y.

Pf. Let $\mathcal{C} = \{\text{all } E \subseteq X \times Y \text{ s.t. cond. holds}\}$. Then, $\mathcal{C}$ is a σ-algebra and contains all rectangles $\Rightarrow \mathcal{C} \supseteq \mathcal{M} \otimes \mathcal{N}$. Details are DIY.

<u>Cor 1.</u> f is $\mathcal{M}\otimes\mathcal{N}$-meas. $\Rightarrow f_x, f^y$ are $\mathcal{N}$- resp. $\mathcal{M}$-meas. for all x, y.

<u>Thm 1.</u> Suppose $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$ are σ-finite. If $E \in \mathcal{M}\otimes\mathcal{N}$, then $x \to \nu(E_x)$, $y \to \mu(E^y)$ are $\mathcal{M}$- resp. $\mathcal{N}$-measurable and

$$(\mu\times\nu)(E) = \int \mu(E^y)\, d\nu(y)$$

$$= \int \nu(E_x)\, d\mu(x).$$